

Arc spaces and DAHA representations

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Abstract

A theorem of Y. Berest, P. Etingof and V. Ginzburg states that finite dimensional irreducible representations of a type A rational Cherednik algebra are classified by one rational number m/n . Every such representation is a representation of the symmetric group S_n . We compare certain multiplicity spaces in its decomposition into irreducible representations of S_n with the spaces of differential forms on a zero-dimensional moduli space associated with the plane curve singularity $x^m = y^n$.

1 Introduction

Rational double affine Hecke algebras (DAHA) were introduced by I. Cherednik ([4]) in his study of the Macdonald conjectures. Their representation theory was extensively studied by C. Dunkl ([6],[7]), Y. Berest, P. Etingof and V. Ginzburg ([1],[2]), I. Gordon and T. Stafford ([11],[12],[13]), M. Varagnolo and E. Vasserot ([21]). Their relation to the geometry of the Hilbert schemes of points and affine Springer fibers was discussed in [13],[21],[22]. We refer the reader to the lectures [8] and the references therein for more complete bibliography.

By construction, rational DAHA $H_{n,c}$ of type A_{n-1} with parameter c has a representation M_c by Dunkl operators in the space of polynomials on the Cartan subalgebra V_n . In [1] it was shown that $H_{n,c}$ has a finite-dimensional representation if and only if $c = m/n$, $m \in (m, n) = 1$. In this case there exists a unique irreducible finite-dimensional representation $L_{m/n}$, which can be constructed as a quotient of $M_{m/n}$ by a certain ideal $I_{m/n}$.

By construction, $L_{m/n}$ carry a natural representation of the symmetric group S_n , so we can split it into irreducible representations of S_n . The

symmetric and antisymmetric parts of $L_{m/n}$ were extensively studied in connection to the representation theory of the spherical DAHA ([1],[21]) and the geometry of the Hilbert scheme of points ([11],[12],[13],[15]). In particular, the space $L_{\frac{n+1}{n}}$ is related to the q, t -Catalan numbers introduced by A. Garsia and M. Haiman ([9]). We are interested in a slightly more general problem of describing the multiplicities of the exterior power $\Lambda^k V_n$ in $L_{m/n}$. Since $\Lambda^k V_n$ is known to be an irreducible representation of S_n , we can describe this multiplicity space as

$$\mathrm{Hom}_{S_n}(\Lambda^k V_n, L_{m/n}).$$

This space is conjectured to be related to some homological invariants of torus knots ([17],[18],[14]). Since

$$\mathrm{Hom}_{S_n}(\Lambda^k V_n, M_{m/n}) \simeq \Omega^k(V_n//S^n),$$

one can ask if the ideal $I_{m/n}$ is related to some natural ideal in $\Omega^k(V_n//S^n)$.

In [10] L. Goettsche, B. Fantechi and D. van Straten constructed a zero-dimensional moduli space $\mathcal{M}_{m,n}$ defined by the coefficients in z -expansion of the equation

$$(1 + z^2 u_2 + z^3 u_3 + \dots + z^n u_n)^m = (1 + z^2 v_2 + z^3 v_3 + \dots + z^m v_m)^n.$$

Our main result is the following

Theorem. *The following isomorphism holds:*

$$\mathrm{Hom}_{S_n}(\Lambda^k V_n, L_{m/n}) \simeq \Omega^k(\mathcal{M}_{m,n}).$$

The proof is explicit: we identify $\mathcal{M}_{m,n}$ with a subscheme in $\mathrm{Spec} \mathbb{C}[u_2, \dots, u_n]$, and identify u_k with the k -th elementary symmetric polynomial on V_n . This allows us to embed $\mathcal{M}_{m,n}$ into the quotient $V_n//S^n$. It rests to compare the defining ideals in both cases.

Corollary 1.1. The left hand side is symmetric in m and n :

$$\mathrm{Hom}_{S_n}(\Lambda^k V_n, L_{m/n}) = \mathrm{Hom}_{S_m}(\Lambda^k V_m, L_{n/m}).$$

Remark 1.2. This relation was proved for $k = 0$ by D. Calaque, B. Enriquez and P. Etingof in [3] by different method. I. Losev announced ([16]) a generalization of their proof for higher values of k .

In sections 2 and 3 we briefly discuss the constructions of the representation $L_{m/n}$ and the moduli space $\mathcal{M}_{m,n}$. In section 4 we compare the constructions and prove the main theorem. In section 5 we discuss the action of Dunkl operators on $\Omega^k(\mathcal{M}_{m,n})$ and the action of the Olshanetsky-Perelomov Hamiltonians on u_l and v_l .

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2 Rational Cherednik algebras

Definition 2.1. ([1]) The rational Cherednik algebra H_c of type A_{n-1} with parameter c is an associative algebra generated by $V = \mathbb{C}^{n-1}, V^*$ and S_n with the following defining relations:

$$\begin{aligned} \sigma \cdot x \cdot \sigma^{-1} &= \sigma(x), \quad \sigma \cdot y \cdot \sigma^{-1} = \sigma(y), \quad \forall x \in V, y \in V^*, \sigma \in S_n \\ x_1 \cdot x_2 &= x_2 \cdot x_1, \quad y_1 \cdot y_2 = y_2 \cdot y_1 \quad \forall x_1, x_2 \in V, y_1, y_2 \in V^* \quad (2.1) \\ y \cdot x - x \cdot y &= \langle y, x \rangle - c \sum_{s \in \mathcal{S}} \langle \alpha_s, x \rangle \langle y, \alpha_s^\vee \rangle \cdot s \quad \forall x \in V, y \in V^*, \end{aligned}$$

where $\mathcal{S} \subset S_n$ is the set of all transpositions, and α_s, α_s^\vee are the corresponding roots and coroots.

The last defining relation is motivated by the following construction of C. Dunkl ([6]).

Definition 2.2. We introduce the Dunkl operators with parameter c by the formula

$$D_i = \frac{\partial}{\partial x_i} - c \sum_{j \neq i} \frac{s_{ij} - 1}{x_i - x_j}.$$

Proposition 2.3. Consider the space $\mathbb{C}[V]$ of polynomial functions on V , where the elements of V act by multiplication and the basis of V^* acts by Dunkl operators. This produces a representation of H_c , i. e. all defining relations (2.1) hold. This representation is denoted by M_c .

An theorem of Y. Berest, P. Etingof, and V. Ginzburg says that for $c > 0$, all finite dimensional irreducible representations of H_c can be obtained from this construction:

Theorem 2.4. ([1]) H_c has finite dimensional representations if and only if $c = m/n$, where m is an integer and $(m, n) = 1$. In this case, $H_{m/n}$ has a unique (up to isomorphism) finite dimensional irreducible representation $L_{m/n}$. For $c = m/n > 0$, $L_c = M_c/I_c$, where I_c is an ideal generated by some homogeneous polynomials of degree m .

Following [7] and [5], we would like to give an explicit construction of these polynomials for $c = m/n$:

$$f_i = \text{Coef}_m[(1 - zx_i)^{-1} \prod_{i=1}^n (1 - zx_i)^{\frac{m}{n}}].$$

Remark 2.5. It is known that f_i form a regular sequence, hence $I_{m/n}$ defines a 0-dimensional complete intersection in V . The Bezout's theorem implies the dimension formula ([1])

$$\dim L_{m/n} = \dim \mathbb{C}[V]/I_{m/n} = m^{n-1}.$$

Let us explain the choice of the polynomials f_i . From now on we will assume that $c = m/n$.

Definition 2.6. Following [7], let us introduce the formal series

$$F(z) = \prod_{i=1}^n (1 - zx_i)^{m/n}, \quad B_i(z) = \frac{1}{1 - zx_i} F(z).$$

Lemma 2.7. ([7]) The action of the Dunkl operators on $B_i(z)$ is given by the formula

$$D_s B_i(z) = \delta_{is} \left(z^2 \frac{dB_i}{dz} + (1 - m) z B_i(z) \right)$$

Corollary 2.8. ([7])

$$D_s \text{Coef}_k[B_i(z)] = \delta_{is} (k - m) \text{Coef}_{k-1}[B_i(z)].$$

In particular,

$$D_s(f_i) = D_s \text{Coef}_m[B_i(z)] = 0 \quad .$$

This explains why f_i generate an ideal invariant under DAHA action.

3 Arc space on a singular curve

In [10] L. Goettsche, B. Fantechi and D. van Straten constructed a zero-dimensional quasihomogeneous complete intersection associated with a curve $\{x^m = y^n\}$. As before, we will assume that $(m, n) = 1$.

Consider a plane curve singularity $C = \{x^m = y^n\}$, and its uniformization $(x, y) = (t^n, t^m)$. Let us consider a general deformation of this parametrization:

$$(x(t), y(t)) = (t^n + u_1 t^{n-1} + \dots + u_n, t^m + v_1 t^{m-1} + \dots + v_m).$$

Consider the ideal $I_{m,n}$ generated by the coefficients in t -expansion of the equation

$$(t^n + u_2 t^{n-2} + \dots + u_n)^m - (t^m + v_2 t^{m-2} + \dots + v_m)^n = 0. \quad (3.1)$$

Remark 3.1. By shifting the parameter t we can annihilate the coefficient u_1 . Since

$$x(t)^m - y(t)^n = (mu_1 - nv_1)t^{mn-1} + \text{terms of lower degree},$$

the equation $u_1 = 0$ implies $v_1 = 0$.

Definition 3.2. The moduli scheme of arcs on C is defined as

$$\mathcal{M}_C = \text{Spec } \mathbb{C}[u_2, \dots, u_n, v_2, \dots, v_m]/I_{m,n}.$$

Lemma 3.3. ([10], Example 1) *The scheme \mathcal{M}_C is a zero-dimensional complete intersection.*

Proof. The equation $x(t)^m - y(t)^n = 0$ is equivalent to the equation

$$mx'(t)y(t) - ny'(t)x(t) = 0. \quad (3.2)$$

The left hand side of (3.2) is a polynomial in t of degree $m+n-3$. Therefore the ideal $I_{m,n}$ is generated by a sequence of the $(m+n-2)$ equations on $(m-1) + (n-1) = (m+n-2)$ variables. Since m and n are coprime, the reduced scheme consists of one point $(x(t), y(t)) = (t^n, t^m)$. \square

Corollary 3.4. The algebra of algebraic differential forms on this moduli space can be described as

$$\Omega^\bullet(\mathcal{M}_C) = \Omega^\bullet(\mathbb{C}^{m+n-2})/(\phi \cdot \omega_1 + d\phi \wedge \omega_2 | \phi \in I_{m,n}).$$

Definition 3.5. We assign the q -grading to u_i and v_i by the formula

$$q(u_i) = q(v_i) = i.$$

Lemma 3.6. ([10]) The multiplicity of \mathcal{M}_C is given by the formula

$$\text{mult}(\mathcal{M}_C) = \frac{(m+n-1)!}{m!n!}$$

Proof. One can check that the ideal $I_{m,n}$ is weighted homogeneous with respect to the q -grading, and the multiplicity of \mathcal{M}_C can be computed using Bezout's theorem:

$$\text{mult}(\mathcal{M}_C) = \frac{2 \cdot 3 \cdot \dots \cdot (m+n-1)}{2 \cdot 3 \cdot \dots \cdot n \cdot 2 \cdot 3 \cdot \dots \cdot m} = \frac{(m+n-1)!}{m!n!}.$$

□

Lemma 3.7. ([10]) The Hilbert series of $\mathbb{C}[u_2, \dots, u_n, v_2, \dots, v_n]/I_{m,n}$ with respect to the q -grading equals to

$$H_{m,n}(q) = \frac{[(m+n-1)!]_q}{[m!]_q[n!]_q} = \prod_{k=2}^n \frac{(1-q^{m+k-1})}{(1-q^k)}.$$

Proof. The ideal $I_{m,n}$ is generated by the weighted homogeneous regular sequence of equations of weights $2, 3, \dots, (m+n-1)$, so the proof is analogous to the Lemma 3.6. □

For the further discussions we have to slightly change the notations. Let us change t to $z = t^{-1}$, then the equation (3.1) will have a form

$$(1 + z^2 u_2 + \dots + z^n u_n)^m = (1 + z^2 v_2 + \dots + z^m v_m)^n. \quad (3.3)$$

Definition 3.8. Let $J_{m/n}$ denote the ideal in $\mathbb{C}[u_1, \dots, u_n]$ generated by the coefficients of the series $(1 + z^2 u_2 + \dots + z^n u_n)^{\frac{m}{n}}$, starting from $m+1$ -st.

Lemma 3.9. Using (3.3), one can express v_i through u_j . The remaining equations on u_i generate the ideal $J_{m/n}$.

Proof. Let us take the n th root of both parts of (3.3):

$$1 + z^2 v_2 + \dots + z^m v_m = (1 + z^2 u_2 + \dots + z^n u_n)^{\frac{m}{n}}.$$

This allows us to express v_i through u_j explicitly, and the remaining equations on u_j express the fact that the right hand side should be a polynomial of degree at most m . □

Corollary 3.10.

$$\mathbb{C}[u_1, \dots, u_n]/J_{m/n} \simeq \mathbb{C}[v_1, \dots, v_m]/J_{n/m}.$$

Example 3.11. For $n = 2$ one can check that $J_{(2k+1)/2}$ is generated by a single polynomial u_2^{k+1} . Therefore the algebra of differential forms $\Omega^\bullet(\mathcal{M}_C)$ is defined by the equations

$$u_2^{k+1} = 0, \quad u_2^k du_2 = 0,$$

and its basis consists of forms

$$1, u_2, \dots, u_2^k, du_2, u_2 du_2, \dots, u_2^{k-1} du_2.$$

Example 3.12. Let $n = 3, m = 4$. The ideal $J_{4/3}$ is generated by the coefficients of the series $(1 + u_2 z^2 + u_3 z^3)^{4/3}$, starting from 5-th, hence its generators are

$$\frac{4}{9}u_2 u_3, \quad \frac{2}{9}u_3^2 - \frac{4}{81}u_2^3.$$

The basis in the quotient is presented by $1, u_2, u_2^2, u_3, u_3^2$.

The algebra $\Omega^\bullet(\mathcal{M}_C)$ is defined by two more equations

$$u_2 du_3 + u_3 du_2 = 0, \quad 2u_3 du_3 - \frac{2}{3}u_2^2 du_2 = 0.$$

Therefore we have 5 one-forms

$$\Omega^1(\mathcal{M}_C) = \langle du_2, du_3, u_2 du_2, u_2 du_3, u_2^2 du_2 \rangle$$

and one two-form $du_2 du_3$.

4 A comparison

It turns out that the ideal $I_{m/n}$ constructed in [7] is related to the moduli space \mathcal{M}_C . Recall that V denotes the standard $(n - 1)$ -dimensional representation of the symmetric group S_n with the coordinates x_i modulo the relation $\sum x_i = 0$.

Definition 4.1. Let us introduce the elementary symmetric polynomials $u_i(x_1, \dots, x_n) \in \mathbb{C}[V]^{S_n}$ by the formula

$$U(z) = \prod_{i=1}^n (1 - zx_i) = 1 + \sum_{i=2}^n z^i u_i(x).$$

Lemma 4.2. *Let us introduce the power sums*

$$p_i = x_1^i + \dots + x_n^i.$$

Then

$$\frac{d}{dz} \ln U(z) = - \sum_{i=0}^{\infty} p_{i+1} z^i.$$

Theorem 4.3. *For $c = m/n$ one have*

$$(I_c)^{S_n} = J_{m/n}.$$

Proof. By construction, the set of generators of the ideal $I_{m/n}$ form a standard representation of S_n (cf. Remark 4.4). Therefore to construct the generators of the symmetric part of $I_{m/n}$, we have to compute

$$[V \otimes \mathbb{C}[V]]^{S_n} / \mathbb{C}[V]_+^{S_n} = \text{Hom}_{S_n}(V, \mathbb{C}[V] / \mathbb{C}[V]_+^{S_n}),$$

where $\mathbb{C}[V]_+^{S_n}$ denotes the ideal generated by the symmetric polynomials of positive degree. It is well known that $\mathbb{C}[V] / \mathbb{C}[V]_+^{S_n}$ is isomorphic to the regular representation of S_n , so there are $(n-1)$ different copies of V in it, generated by x_i^k for $1 \leq k \leq n-1$. This means that the symmetric part of the ideal $I_{m/n}$ is generated by $(n-1)$ polynomials:

$$(I_c)^{S_n} = \langle \sum_i x_i^k f_i \mid 1 \leq k \leq n-1 \rangle.$$

Let us compute these generators.

Recall that

$$F(z) = \prod_{i=1}^n (1 - zx_i)^{m/n} = \sum_{k=0}^{\infty} v_k(x) z^k.$$

By Lemma 4.2

$$\frac{d}{dz} F(z) = \frac{m}{n} F(z) \frac{d}{dz} \ln U(z) = -\frac{m}{n} F(z) \cdot \sum_{i=0}^{\infty} p_{i+1} z^i,$$

Therefore

$$(k+1)v_{k+1} = -\frac{m}{n} \sum_{i=0}^k v_{k-i} p_{i+1}. \quad (4.1)$$

On the other hand,

$$\sum_i f_i x_i^k = \sum_i \text{Coef}_m \frac{x_i^k}{1 - zx_i} F(z) = \sum_{j=0}^m v_{m-j} p_{j+k}.$$

Therefore by (4.1)

$$\begin{aligned} \sum_i f_i x_i &= \sum_{j=0}^m v_{m-j} p_{j+1} = -\frac{n}{m}(m+1)v_{m+1}, \\ \sum_i f_i x_i^2 &= \sum_{j=0}^m v_{m-j} p_{j+2} = -\frac{n}{m}(m+2)v_{m+2} - v_{m+1}p_1, \dots \\ \sum_i f_i x_i^k &= \sum_{j=0}^m v_{m-j} p_{j+k} = -\frac{n}{m}(m+k)v_{m+k} - \sum_{j=1}^{k-1} v_{m+j} p_{k-j}, \end{aligned}$$

and $v_{m+1}, \dots, v_{m+n-1}$ can be obtained from $\sum_i f_i x_i^k$ with a triangular change of variables.

It rests to note that by Lemma 3.9 the polynomials $v_{m+1}, \dots, v_{m+n-1}$ generate the ideal $J_{m/n}$. □

Remark 4.4. If we plug in $k = m - 1$ in (4.1), we will get $mv_m = -\frac{m}{n} \sum_{i=0}^{m-1} v_{k-i} p_{i+1}$, hence $nv_m + \sum_{i=0}^{m-1} v_{k-i} p_{i+1} = 0$, and

$$\sum_j f_j = \sum_j \text{Coef}_m [(1 - zx_j)^{-1} F(z)] = nv_m + \sum_{i=1}^m v_{m-i} p_i = 0.$$

Corollary 4.5.

$$\text{Coef}_k[B_i(z)] \in I_{m/n} \quad \text{for } k \geq m.$$

Proof. We have

$$\begin{aligned} \text{Coef}_k[B_i(z)] &= \text{Coef}_k \left[\frac{F(z)}{1 - zx_i} \right] = \sum_{a=0}^k x_i^a v_{k-a} = \\ &= \sum_{a=0}^{k-m-1} x_i^a v_{k-a} + \sum_{a=k-m}^k x_i^a v_{k-a}. \end{aligned}$$

By Theorem 4.3 the first sum belongs to $I_{m/n}$, and the second sum can be rewritten as

$$\sum_{a=0}^m x_i^{k-m+a} v_{m-a} = x_i^{k-m} \sum_{a=0}^m x_i^a v_{m-a} = x_i^{k-m} \text{Coef}_m[B_i(z)] \in I_{m/n}.$$

□

Lemma 4.6. *Let $h(x_1, \dots, x_n)$ be a symmetric function in x_2, \dots, x_n . There exist functions $\phi_1, \dots, \phi_{n-1}, \rho_1, \dots, \rho_{n-1} \in \mathbb{C}[V]^{S_n}$ such that*

$$h \cdot f_1 = \sum_{k>m} (\phi_j \frac{\partial v_k}{\partial x_1} + \rho_j v_k). \quad (4.2)$$

Proof. One can present h as a linear combination of some powers of x_1 multiplied by some symmetric polynomials in x_2, \dots, x_n . Therefore it is sufficient to prove (4.2) for $h = x_1^k$.

Since

$$(1 - zx_1) \frac{\partial F(z)}{\partial x_1} = -\frac{m}{n} z F(z),$$

one has

$$x_1 \frac{\partial v_k}{\partial x_1} = \frac{\partial v_{k+1}}{\partial x_1} + \frac{m}{n} v_k.$$

Using this equation, one can express

$$x_1^k f_1 = -\frac{n}{m} x_1^k \frac{\partial v_{m+1}}{\partial x_1}$$

via v_k and $\frac{\partial v_k}{\partial x_1}$ with $k > m$. □

Theorem 4.7.

$$\text{Hom}_{S_n}(\Lambda^k V, L_{m/n}) \simeq \Omega^k(\mathcal{M}_{m,n}).$$

Remark that

$$\text{Hom}_{S_n}(\Lambda^k V, M_{m/n}) = \text{Hom}_{S_n}(\Lambda^k V, \mathbb{C}[V]) = \Omega^k(V)^{S_n} \simeq \Omega^k(V//S_n),$$

where the last isomorphism follows from the results of [20]: every S_n -invariant differential form on V can be obtained as a pullback of a form on $V//S_n$. To fix the notation, we give the following

Definition 4.8. We introduce a map

$$\lambda : \Omega^k(V//S_n) \rightarrow \text{Hom}_{S_n}(\Lambda^k V, \mathbb{C}[V])$$

by the formula

$$\lambda_{i_1, \dots, i_k}(\omega) = \pi^* \omega \left(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_k}} \right),$$

where $\pi : V \rightarrow V//S_n$ denotes the natural projection.

Proposition 4.9. *The following equation holds:*

$$\lambda_{i_1, \dots, i_s}(du_{k_1} \wedge \dots \wedge du_{k_s}) = \left| \frac{\partial u_{k_a}}{\partial x_{i_b}} \right|.$$

Proof of Theorem 4.7. Let us check that the map λ sends the defining equations of $\Omega^\bullet(\mathcal{M}_C)$ inside the ideal $I_{m/n}$.

Recall that

$$F(z) = \prod_{i=1}^n (1 - zx_i)^{m/n} = \sum_{k=0}^{\infty} v_k(u) z^k,$$

and the defining ideal of $\Omega^\bullet(\mathcal{M}_C)$ is generated by the equations

$$v_k(u) = dv_k(u) = 0 \quad \text{for } k > m$$

We checked in Theorem 4.3 that $\lambda(v_k) \in I_{m/n}$, let us check that $\lambda(dv_k) \in I_{m/n}$.

Remark that

$$\lambda_i(dv_k) = \frac{\partial v_k}{\partial x_i} = \text{Coef}_k \frac{\partial F(z)}{\partial x_i} = -\frac{m}{n} \text{Coef}_{k-1} \left[\frac{F(z)}{1 - zx_i} \right].$$

This coefficient belongs to $I_{m/n}$ by Corollary 4.5.

Similarly to the proof of Theorem 4.3 one can check that every element of $\text{Hom}_{S_n}(\Lambda^k V, I_{m/n})$ can be presented as a combination of the determinants of the form

$$\begin{vmatrix} f_{\alpha_1} h_1 & f_{\alpha_2} h_2 & \dots & f_{\alpha_k} h_k \\ \frac{\partial u_{\beta_1}}{\partial x_{\alpha_1}} & \frac{\partial u_{\beta_1}}{\partial x_{\alpha_2}} & \dots & \frac{\partial u_{\beta_1}}{\partial x_{\alpha_k}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_{\beta_{k-1}}}{\partial x_{\alpha_1}} & \frac{\partial u_{\beta_{k-1}}}{\partial x_{\alpha_2}} & \dots & \frac{\partial u_{\beta_{k-1}}}{\partial x_{\alpha_k}} \end{vmatrix}$$

with symmetric coefficients, where h_i are symmetric in all variables but x_{α_i} . By Lemma 4.6, we can present (modulo $J_{m/n}$) every such determinant as a combination of the expressions

$$\lambda_{\alpha_1, \dots, \alpha_k}(dv_s \wedge du_{\beta_1} \wedge \dots \wedge du_{\beta_{k-1}}), \quad s > m$$

with symmetric coefficients. It rests to note that the form

$$dv_s \wedge du_{\beta_1} \wedge \dots \wedge du_{\beta_{k-1}} = dv_s \wedge \omega$$

belongs to the defining ideal of $\Omega^\bullet(\mathcal{M}_C)$. \square

5 Action of Dunkl operators

We start with the following reformulation of the Corollary 2.8.

Lemma 5.1.

$$D_s \lambda_i(dv_k) = \delta_{is}(k-1-m) \lambda_i(dv_{k-1}).$$

Proof. Remark that

$$\lambda_i(dv_k) = \frac{\partial v_k}{\partial x_i} = -\frac{m}{n} \text{Coef}_{k-1}[B_i(z)],$$

by Corollary 2.8

$$D_s \text{Coef}_{k-1}[B_i(z)] = \delta_{is}(k-1-m) \text{Coef}_{k-2}[B_i(z)].$$

hence

$$D_s \lambda_i(dv_k) = \delta_{is}(k-1-m) \lambda_i(dv_{k-1}).$$

\square

Lemma 5.2. ([7]) *The following product rule holds for Dunkl operators:*

$$D_i(fg) = D_i(f)g + D_i(g)f + \frac{m}{n} \sum_{i \neq j} \frac{(f - s_{ij}(f))(g - s_{ij}(g))}{x_i - x_j}.$$

Corollary 5.3. If g is a symmetric polynomial then

$$D_i(fg) = D_i(f)g + D_i(g)f.$$

Lemma 5.4. ([7]) Suppose f_1, \dots, f_m are polynomials satisfying $(ij)f_l = f_l$ if $l \notin \{i, j\}$. Then

$$D_i(f_1 \cdots f_m) = \sum_{l=1}^m (D_i f_l) \prod_{s \neq l} f_s + \frac{m}{n} \sum_{l \neq i} \frac{(f_i - s_{il}(f_i))(f_l - s_{il}(f_l))}{x_i - x_l} \prod_{s \neq l, i} f_s. \quad (5.1)$$

Lemma 5.5. Suppose that the functions $a_j, 1 \leq j \leq k$ are symmetric with respect to all variables but x_1 . Consider a matrix $M = (s_{1i}(a_j))_{j=1}^k$. Then

$$D_i \det(M) = \sum_l \det(M_{i,l}), \quad (5.2)$$

where $M_{s,l}$ denotes the matrix M where the entries in the l 'th row are replaced by their images under D_i .

Remark 5.6. This lemma shows that although D_i is not a first order differential operator, it acts on these determinants as a first order differential operator would act.

Proof. Let us expand $\det(M)$ and apply the equation (5.1). We have to show that the "correction terms" with divided differences will cancel out. These terms are labelled by the pairs (σ, l) where $l \neq i$ and $\sigma \in S_k$, and the terms corresponding to (σ, l) and $((il)\sigma, l)$ have opposite sign but same value

$$\begin{aligned} & \frac{1}{x_i - x_l} (s_{1i}a_{\sigma(i)} - s_{il}s_{1i}a_{\sigma(i)}) (s_{1l}a_{\sigma(l)} - s_{il}s_{1l}a_{\sigma(l)}) = \\ & \frac{1}{x_i - x_l} (s_{1i}a_{\sigma(i)} - s_{1l}a_{\sigma(i)}) (s_{1l}a_{\sigma(l)} - s_{1i}a_{\sigma(l)}). \end{aligned}$$

□

We are ready to describe the action of Dunkl operators on the image of the map λ . By Lemma 5.3 and it is sufficient to compute the action of D_i on the components of the differential form $dv_{\alpha_1} \wedge \dots \wedge dv_{\alpha_k}$.

Theorem 5.7. Suppose that $\beta_1 < \dots < \beta_j$. If $\beta_1 > 1$, then

$$D_1 \left| \frac{\partial v_{\alpha_i}}{\partial x_{\beta_j}} \right| = 0.$$

If $\beta_1 = 1$, then

$$D_1 \left| \frac{\partial v_{\alpha_i}}{\partial x_{\beta_j}} \right| = \begin{vmatrix} (\alpha_1 - 1 - m) \frac{\partial v_{\alpha_1-1}}{\partial x_1} & \cdots & (\alpha_k - 1 - m) \frac{\partial v_{\alpha_k-1}}{\partial x_1} \\ \frac{\partial v_{\alpha_1}}{\partial x_{\beta_2}} & \cdots & \frac{\partial v_{\alpha_k}}{\partial x_{\beta_2}} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_{\alpha_1}}{\partial x_{\beta_k}} & \cdots & \frac{\partial v_{\alpha_k}}{\partial x_{\beta_k}} \end{vmatrix}$$

Proof. Follows from Lemma 5.5 and Lemma 5.1. \square

Definition 5.8. ([19],[8]) The quantum Olshanetsky-Perelomov Hamiltonians are defined as

$$H_k = \sum_{s=1}^n D_s^k.$$

Lemma 5.9.

$$\sum_i \lambda_i(dv_k) = (k - 1 - m)v_{k-1}$$

Proof.

$$\begin{aligned} \sum_i \lambda_i(dv_k) &= \sum_i \frac{\partial F(z)}{\partial x_s} = -\frac{m}{n} z F(z) \sum_i \frac{1}{1 - zx_i} = \\ &= -mzF(z) - \frac{m}{n} z^2 F(z) \sum_i \frac{x_i}{1 - zx_i} = -mzF(z) + z^2 \frac{dF(z)}{dz}. \end{aligned}$$

\square

Theorem 5.10. *The action of Hamiltonians on u_l and v_l has the following form:*

$$\begin{aligned} H_k(v_l) &= (l - 1 - m) \cdots (l - k + m) v_{l-k}, \\ H_k(u_l) &= \left(\frac{m}{n}\right)^{k-1} (l - 1 - n) \cdots (l - k - n) u_{l-k} \end{aligned} \quad (5.3)$$

Proof. Since u_l is symmetric, $D_i(u_l) = \frac{\partial u_l}{\partial x_i}$, and one can check that

$$D_i^2(u_l) = \frac{m}{n} (l - 1 - n) \frac{\partial u_{l-1}}{\partial x_i}, \quad \sum_i \lambda_i(du_l) = (l - 1 - n) u_{l-1}$$

hence

$$H_k(u_l) = \left(\frac{m}{n}\right)^{k-1} (l-1-n) \cdots (l-k+1-n) \sum_i \frac{\partial u_{l-k+1}}{\partial x_i} =$$

$$\left(\frac{m}{n}\right)^{k-1} (l-1-n) \cdots (l-k+1-n) (l-k-n) u_{l-k}.$$

The proof for v_l is similar – it follows from Lemma 5.1 and Lemma 5.9. \square

References

- [1] Y. Berest, P. Etingof, V. Ginzburg. Finite-dimensional representations of rational Cherednik algebras. *Int. Math. Res. Not.* 2003, no. 19, 1053–1088.
- [2] Y. Berest, P. Etingof, V. Ginzburg. Morita equivalence of Cherednik algebras. *J. Reine Angew. Math.* **568** (2004), 81–98.
- [3] D. Calaque, B. Enriquez, P. Etingof. Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, 165–266, *Progr. Math.*, 269, Birkhäuser Boston, Inc., Boston, MA, 2009.
- [4] I. Cherednik. A unification of Knizhnik-Zamolodchikov and Dunkl operators via affine Hecke algebras. *Invent. Math.* **106** (1991), 411–431.
- [5] T. Chmutova, P. Etingof. On some representations of the rational Cherednik algebra. *Representation theory* **7** (2003), 641–650.
- [6] C. Dunkl. Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.* **311** (1989), 167–183.
- [7] C. Dunkl. Intertwining operators and polynomials associated with the symmetric group. *Monatsh. Math.* **126** (1998), no. 3, 181–209.
- [8] P. Etingof, X. Ma. Lecture notes on Cherednik algebras. *arXiv*: 1001.0432
- [9] A. Garsia, M. Haiman. A remarkable q, t -Catalan sequence and q -Lagrange inversion. *J. Algebraic Combin.* **5** (1996), no. 3, 191–244.

- [10] L. Goettsche, B. Fantechi, D. van Straten. Euler number of the compactified Jacobian and multiplicity of rational curves. *J. Algebraic Geom.* **8** (1999), no. 1, 115–133.
- [11] I. Gordon. On the quotient ring by diagonal invariants. *Invent. Math.* **153** (2003), no. 3, 503–518.
- [12] I. Gordon, T. Stafford. Rational Cherednik algebras and Hilbert schemes. *Adv. Math.* **198** (2005), no. 1, 222–274.
- [13] I. Gordon, T. Stafford. Rational Cherednik algebras and Hilbert schemes. II. Representations and sheaves. *Duke Math. J.* **132** (2006), no. 1, 73–135.
- [14] E. Gorsky, A. Oblomkov, J. Rasmussen, V. Shende. Rational DAHA and the homology of torus knots. In preparation.
- [15] M. Haiman. Vanishing theorems and character formulas for the Hilbert scheme of points in the plane. *Invent. Math.* **149** (2002), no. 2, 371–407.
- [16] I. Losev. In preparation.
- [17] A. Oblomkov, V. Shende. The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link. arXiv: 1003.1568
- [18] A. Oblomkov, J. Rasmussen, V. Shende. The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link. In preparation.
- [19] M. Olshanetsky, A. Perelomov. Quantum integrable systems related to Lie algebras. *Phys. Rep.* **94** (1983), no.6, 313–404.
- [20] L. Solomon, Invariants of finite reflection groups. *Nagoya Mathem. J.* **22** (1963), 57–64.
- [21] M. Varagnolo, E. Vasserot. Finite-dimensional representations of DAHA and affine Springer fibers: the spherical case. *Duke Math. J.* **147** (2009), no. 3, 439–540.
- [22] Z. Yun. Towards a Global Springer Theory II: the double affine action. arXiv:0904.3371

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